Résolution numérique d'une équation différentielle
Méthode de RUNGE-KUTTA RK4

Considérons une équation différentielle du premier ordre :

$$\frac{dy}{dx} = f(x,y)$$

La méthode RK4 utilise **plusieurs points intermédiaires** pour calculer la valeur de y_{i+1} à partir de la valeur de y_i :

On considère un point intermédiaire A d'abscisse x_i+h/2 dont la valeur de l'ordonnée est donnée par :

$$y_{iA} = y_i + \left(\frac{dy}{dx}\right)_i \times \frac{h}{2} \quad \text{soit} \quad y_{iA} - y_i = \left(\frac{dy}{dx}\right)_i \times \frac{h}{2} = \frac{k_1}{2}$$

puis un point B d'ordonnée :

$$y_{iB} = y_i + \left(\frac{dy}{dx}\right)_{iA} \times \frac{h}{2} \quad \text{soit} \quad y_{iB} - y_i = \left(\frac{dy}{dx}\right)_{iA} \times \frac{h}{2} = \frac{k_2}{2}$$

On calcule alors l'ordonnée d'un point C d'abscisse x_i+h à l'aide de la relation :

$$y_{iC} = y_i + \left(\frac{dy}{dx}\right)_{iB} \times h \quad \text{soit} \quad y_{iC} - y_i = \left(\frac{dy}{dx}\right)_{iB} \times h = k_3$$

Soit $\left(\frac{dy}{dx}\right)_{ic}$ la valeur de $\left(\frac{dy}{dx}\right)$ au point C.

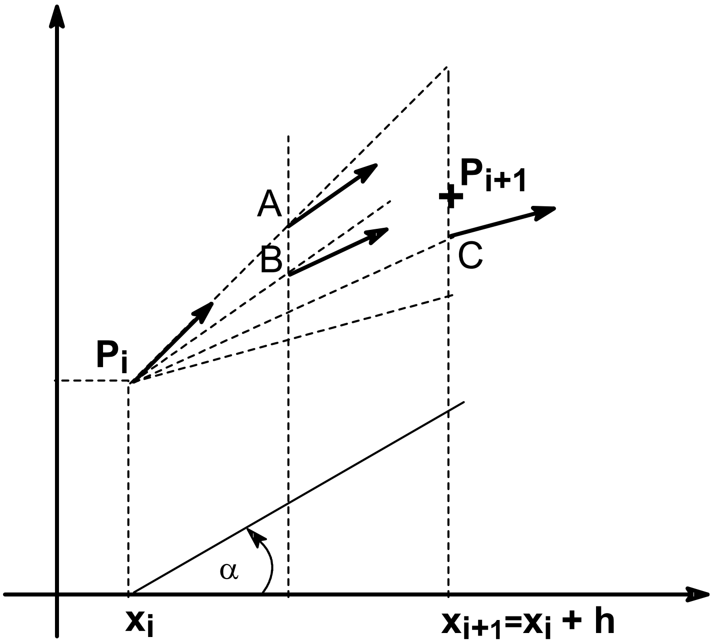
On pose : $$\left(\frac{dy}{dx}\right)_{ic} \times h = k_4$$

L'ordonnée définitive y_{i+1} du point d'abscisse x_i+h est donnée par la relation :

$$y_{i+1} = y_i + \frac{1}{6}\left[\left(\frac{dy}{dx}\right)_i + 2\times\left(\frac{dy}{dx}\right)_{iA} + 2\times\left(\frac{dy}{dx}\right)_{iB} + \left(\frac{dy}{dx}\right)_{iC}\right]\times h$$

ou

$$y_{i+1} = y_i + \frac{1}{6}\left[k_1 + 2\times k_2 + 2\times k_3 + k_4\right]$$



$$tg\alpha = \frac{1}{6}\left[\left(\frac{dy}{dx}\right)_i + 2\times\left(\frac{dy}{dx}\right)_{iA} + 2\times\left(\frac{dy}{dx}\right)_{iB} + \left(\frac{dy}{dx}\right)_{iC}\right]$$

Présentation géométrique de la méthode RK4

La méthode de RUNGE KUTTA d'ordre 4 définit deux suites, h étant le pas de discrétisation en x :

- une première qui permet de définir les valeurs de x
  - terme initial : x_0
  - relation de récurrence : $x_{i+1} = x_i+h$
- une deuxième qui permet d'évaluer les valeurs de y
  - terme initial : y_0
  - relation de récurrence : $y_{i+1} = y_i+1/6(k_1+2k_2+2k_3+k_4)$
    avec
    $k_1 = h\times f(x_i,y_i)$
    $k_2 = h\times f(x_i+h/2,y_i+k_1/2)$
    $k_3 = h\times f(x_i+h/2,y_i+k_2/2)$
    $k_4 = h\times f(x_i+h,y_i+k_3)$

Exemple : $\dfrac{dy}{dx} = -2x.y$

P_i	$\begin{cases} x_i \\ y_i \end{cases}$	$\left(\dfrac{dy}{dx}\right)_i = -2 \times x_i \times y_i = \dfrac{k_1}{dx}$
A	$\begin{cases} x_i + \dfrac{dx}{2} \\ y_i + \left(\dfrac{dy}{dx}\right)_i \times \dfrac{dx}{2} = y_i + \dfrac{k_1}{2} \end{cases}$	$\left(\dfrac{dy}{dx}\right)_{ia} = -2 \times \left(x_i + \dfrac{dx}{2}\right) \times \left(y_i + \dfrac{k_1}{2}\right) = \dfrac{k_2}{dx}$
B	$\begin{cases} x_i + \dfrac{dx}{2} \\ y_i + \left(\dfrac{dy}{dx}\right)_{iA} \times \dfrac{dx}{2} = y_i + \dfrac{k_2}{2} \end{cases}$	$\left(\dfrac{dy}{dx}\right)_{iB} = -2 \times \left(x_i + \dfrac{dx}{2}\right) \times \left(y_i + \dfrac{k_2}{2}\right) = \dfrac{k_3}{dx}$
C	$\begin{cases} x_i + dx \\ y_i + \left(\dfrac{dy}{dx}\right)_{iB} \times dx = y_i + k_3 \end{cases}$	$\left(\dfrac{dy}{dx}\right)_{iC} = -2 \times (x_i + dx) \times (y_i + k_3) = \dfrac{k_4}{dx}$
P_{i+1}	$y_{i+1} = y_i + \dfrac{1}{6}\left[\left(\dfrac{dy}{dx}\right)_i + 2 \times \left(\dfrac{dy}{dx}\right)_{iA} + 2 \times \left(\dfrac{dy}{dx}\right)_{iB} + \left(\dfrac{dy}{dx}\right)_{iC}\right] \times dx$ $y_{i+1} = y_i + \dfrac{1}{6}\left[k_1 + 2 \times k_2 + 2 \times k_3 + k_4\right]$	

Exemple mathématique : considérons l'équation différentielle : $\frac{dy}{dx} = -2x.y$ avec $y_{(x=0)}=1$

La solution analytique est : $y = \exp(-x^2)$

i	x_i	y_i
0	0	1
1	0,1	0,9900
2	0,2	0,9608
3	0,3	0,9139
4	0,4	0,8521
5	0,5	0,7788

Méthode RK4 :

i	x_i	y_i	$k_1=-2 \times x_i \times y_i \times \Delta x$	$k_2 = -2 \times (x_i+\Delta x/2)\times(y_i + k_1/2)\times\Delta x$
0	0	1	$-2 \times 0 \times 1 \times 0,1=0$	$-2 \times 0,05 \times 1 \times 0,1=-0,01$
1	0,1	0.990	$-2 \times 0,1 \times 0,99 \times 0,1=0,0198$	$-2 \times 0,15 \times 0,98 \times 0,1=-0,0294$
2	0,2			
3	0,2			
4	0,4			
5	0,5			

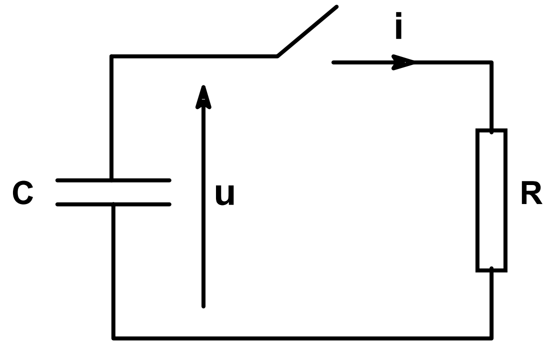
i	x_i	y_i	$k_3 = -2 \times (x_i+\Delta x/2)\times(y_i + k_2/2)\times\Delta x$	$k_4 = -2 \times (x_i+\Delta x)\times(y_i + k_3)\times\Delta x$
0	0	1	$-2 \times 0,05 \times 0,995\times 0,1=-9,9510^{-3}$	$-2 \times 0,1 \times 0,995 \times 0,1= -0,0199$
1	0,1	0,990	$-2 \times 0,15 \times 0,975\times 0,1=-2,92610^{-2}$	$-2 \times 0,2 \times 0,961 \times 0,1= -0,0384$
2	0,2	0,961		
3	0,2			
4	0,4			
5	0,5			

Avec EXCEL :

	A	B	C
	x	k1	k2
6	0	=-2*x*y*dx	=-2*(x+dx/2)*(y+k1/2)*dx
7	=A6+dx	=-2*x*y*dx	=-2*(x+dx/2)*(y+k1/2)*dx
8	=A7+dx	=-2*x*y*dx	=-2*(x+dx/2)*(y+k1/2)*dx

	D	E	F
	k3	k4	y
6	=-2*(x+dx/2)*(y+k2/2)*dx	=-2*(x+dx)*(y+k3)*dx	1
7	=-2*(x+dx/2)*(y+k2/2)*dx	=-2*(x+dx)*(y+k3)*dx	=F6+1/6*(B6+2*C6+2*D6+E6)
8	=-2*(x+dx/2)*(y+k2/2)*dx	=-2*(x+dx)*(y+k3)*dx	=F7+1/6*(B7+2*C7+2*D7+E7)

Exercice RK4_1 : décharge d'un condensateur dans une résistance.



$$u = Ri = -R\frac{dq}{dt} = -RC\frac{du}{dt}$$

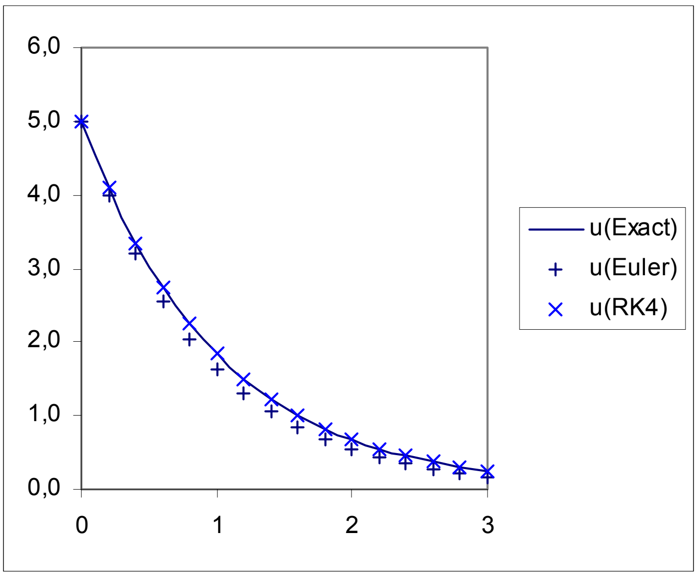
$$\frac{du}{dt} = -\frac{1}{RC}u$$

Exercice : Utiliser la méthode RK4 pour résoudre cette équation différentielle :

$$k_1 = h \times f(x_i, y_i)$$
$$k_2 = h \times f(x_i + h/2, y_i + k_1/2)$$
$$k_3 = h \times f(x_i + h/2, y_i + k_2/2)$$
$$k_4 = h \times f(x_i + h, y_i + k_3)$$

$k_1 = \left(-\frac{1}{RC}u_i\right)dt$	$k_2 = \left(-\frac{1}{RC}\left[u_i + \frac{k_1}{2}\right]\right)dt$
$k_3 = \left(-\frac{1}{RC}\left[u_i + \frac{k_2}{2}\right]\right)dt$	$k_4 = \left(-\frac{1}{RC}\left[u_i + k_3\right]\right)dt$

$$u_{t+dt} = u_t + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$



Détail de la feuille de calcul

	A	B	C	D
6	**t**	**k1**	**k2**	**k3**
7	0	=-dt/RC*F7	=-dt/RC*(F7+k1_/2)	=-dt/RC*(F7+k2_/2)
8	=A7+dt	=-dt/RC*F8	=-dt/RC*(F8+k1_/2)	=-dt/RC*(F8+k2_/2)
9	=A8+dt	=-dt/RC*F9	=-dt/RC*(F9+k1_/2)	=-dt/RC*(F9+k2_/2)
10	=A9+dt	=-dt/RC*F10	=-dt/RC*(F10+k1_/2)	=-dt/RC*(F10+k2_/2)
11	=A10+dt	=-dt/RC*F11	=-dt/RC*(F11+k1_/2)	=-dt/RC*(F11+k2_/2)
12	=A11+dt	=-dt/RC*F12	=-dt/RC*(F12+k1_/2)	=-dt/RC*(F12+k2_/2)
13	=A12+dt	=-dt/RC*F13	=-dt/RC*(F13+k1_/2)	=-dt/RC*(F13+k2_/2)
14	=A13+dt	=-dt/RC*F14	=-dt/RC*(F14+k1_/2)	=-dt/RC*(F14+k2_/2)
15	=A14+dt	=-dt/RC*F15	=-dt/RC*(F15+k1_/2)	=-dt/RC*(F15+k2_/2)
16	=A15+dt	=-dt/RC*F16	=-dt/RC*(F16+k1_/2)	=-dt/RC*(F16+k2_/2)
17	=A16+dt	=-dt/RC*F17	=-dt/RC*(F17+k1_/2)	=-dt/RC*(F17+k2_/2)
18	=A17+dt	=-dt/RC*F18	=-dt/RC*(F18+k1_/2)	=-dt/RC*(F18+k2_/2)

	E	F
6	**k4**	**u(RK4)**
7	=-dt/RC*(F7+k3_)	**5**
8	=-dt/RC*(F8+k3_)	=F7+1/6*(B7+2*C7+2*D7+E7)
9	=-dt/RC*(F9+k3_)	=F8+1/6*(B8+2*C8+2*D8+E8)
10	=-dt/RC*(F10+k3_)	=F9+1/6*(B9+2*C9+2*D9+E9)
11	=-dt/RC*(F11+k3_)	=F10+1/6*(B10+2*C10+2*D10+E10)
12	=-dt/RC*(F12+k3_)	=F11+1/6*(B11+2*C11+2*D11+E11)
13	=-dt/RC*(F13+k3_)	=F12+1/6*(B12+2*C12+2*D12+E12)
14	=-dt/RC*(F14+k3_)	=F13+1/6*(B13+2*C13+2*D13+E13)
15	=-dt/RC*(F15+k3_)	=F14+1/6*(B14+2*C14+2*D14+E14)
16	=-dt/RC*(F16+k3_)	=F15+1/6*(B15+2*C15+2*D15+E15)
17	=-dt/RC*(F17+k3_)	=F16+1/6*(B16+2*C16+2*D16+E16)
18	=-dt/RC*(F18+k3_)	=F17+1/6*(B17+2*C17+2*D17+E17)

dt	0,1	s
RC	0,5	s
U_0	5	V

Exercice RK4 2 : Cinétique chimique

Considérons la réaction A + B → P

	A	B	P
Conc. initiales	c	c	0
C(t)	c-y	c-y	y

Supposons la réaction d'ordre global 2 (1 par rapport à A et 1 par rapport à B) :

$$\frac{dy}{dt} = k \times (c - y)^2$$

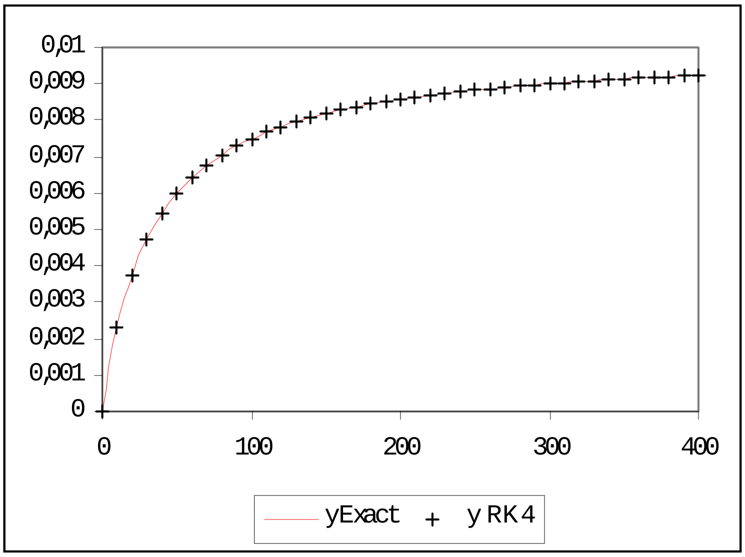
La solution analytique est :

$$y = c \times \left(1 - \frac{1}{k \times t \times c + 1}\right)$$

On pourra donc confronter les valeurs obtenues par la méthode analytique et celles obtenus par les méthodes numériques.

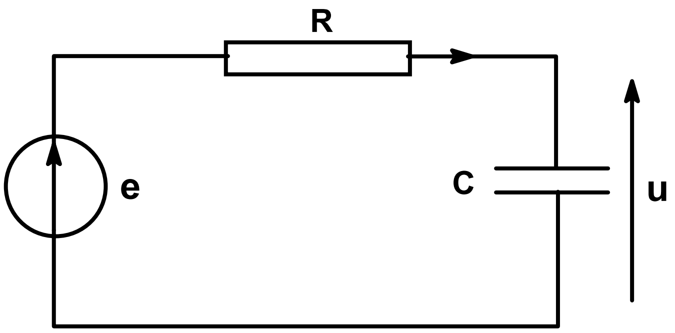
$k_1 = dt \times k(c - y_t)^2$	$k_2 = dt \times k \times \left(c - y_t - \frac{k_1}{2}\right)^2$
$k_3 = dt \times k \times \left(c - y_t - \frac{k_2}{2}\right)^2$	$k_4 = dt \times k \times (c - y_t - k_3)^2$

$$y_{t+dt} = y_t + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$



h	k	c_0
10	3	0,01
s	$L.mol^{-1}s^{-1}$	$mol.L^{-1}$

Exercice RK4 3 : Circuit RC alimenté par une tension e(t).



$$\frac{du}{dt} = \frac{1}{RC}(e - u)$$

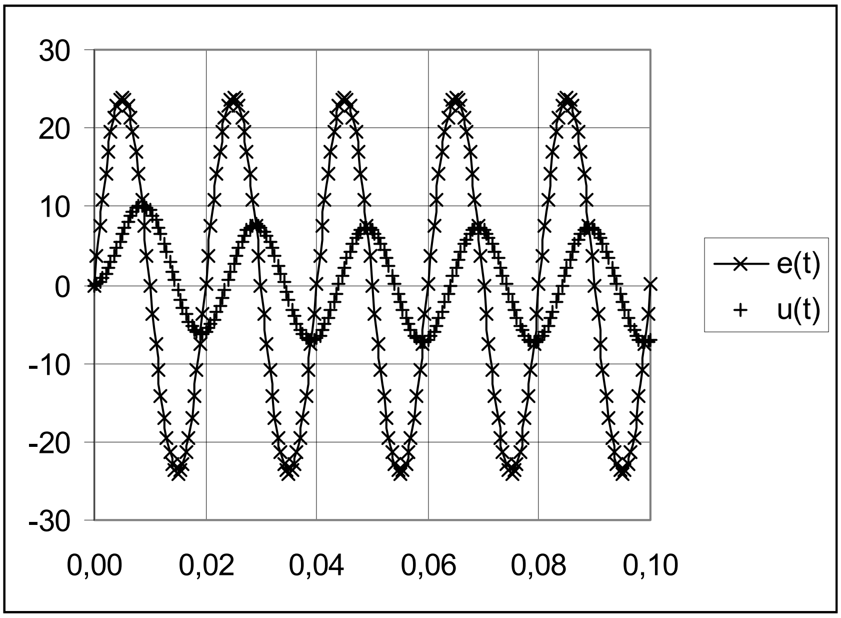
$k_1 = dt \times \frac{1}{RC}[e_t - u_t]$	$k_2 = dt \times \frac{1}{RC} \times \left(e_{t+dt/2} - u_t - \frac{k_1}{2}\right)$
$k_3 = dt \times \frac{1}{RC} \times \left(e_{t+dt/2} - u_t - \frac{k_2}{2}\right)$	$k_4 = dt \times \frac{1}{RC} \times (e_{t+dt} - u_t - k_3)$

$$u_{t+dt} = u_t + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

On retrouvera facilement ces expressions des k_i en considérant que les valeurs de k_2 et k_3 se rapportent à la date t_i +dt/2.

e = $24 \times \sin(2\pi ft)$

R(ohm)	$1{,}0\times10^4$		f (Hz)	50		RC	$1{,}0\times10^{-2}$
C(F)	$1{,}0\times10^{-6}$		dt(s)	$5{,}0\times10^{-4}$			



On observe le régime transitoire initial.

Détail de la feuille de calcul :

	A	B	C	D	E
	t	e(t)	u(t)	Δu	k1
6	0	=(24*SIN(2*PI()*f*t))	0	(k1+2*k2+2*k3+k4)/6	=dt*1/RC*(B6-C6)
7	=A6+dt	=(24*SIN(2*PI()*f*t))	=C6+D6	(k1+2*k2+2*k3+k4)/6	=dt*1/RC*(B7-C7)
8	=A7+dt	=(24*SIN(2*PI()*f*t))	=C7+D7	(k1+2*k2+2*k3+k4)/6	=dt*1/RC*(B8-C8)

	F	G	H	I
	k2	k3	k4	e(t+dt/2)
6	=dt/RC*(I6-C6-k1/2)	=dt/RC*(I6-C6-k2/2)	=dt/RC*(B7-C6-k3)	=(24*SIN(2*PI()*f*(t+dt/2)))
7	=dt/RC*(I7-C7-k1/2)	=dt/RC*(I7-C7-k2/2)	=dt/RC*(B8-C7-k3)	=(24*SIN(2*PI()*f*(t+dt/2)))
8	=dt/RC*(I8-C8-k1/2)	=dt/RC*(I8-C8-k2/2)	=dt/RC*(B9-C8-k3)	=(24*SIN(2*PI()*f*(t+dt/2)))

Exercice RK4 4 : Pendule pesant

$$\theta'' + \omega_0^2 \sin\theta = 0$$

On pose : $\frac{d\theta}{dt} = \Omega$

L'équation ci-dessus devient :

$$\frac{d^2\theta}{dt^2} = \frac{d\Omega}{dt} = -\omega_0^2 \sin\theta$$

Méthode RK4 :

P_i	$\begin{cases} t_i \\ \theta_i \\ \Omega_i \end{cases}$ $\left(\frac{d\theta}{dt}\right)_i = \Omega_i = \frac{j_1}{h}$ $\left(\frac{d\Omega}{dt}\right)_i = -\omega_0^2 \times \sin\theta_i = \frac{k_1}{h}$
A	$\begin{cases} t_i + \frac{h}{2} \\ \theta_i + \left(\frac{d\theta}{dt}\right)_i \times \frac{h}{2} = \theta_i + \frac{j_1}{2} \\ \Omega_i + \left(\frac{d\Omega}{dt}\right)_i \times \frac{h}{2} = \Omega_i + \frac{k_1}{2} \end{cases}$ $\left(\frac{d\theta}{dt}\right)_{ia} = \Omega_i + \frac{k_1}{2} = \frac{j_2}{h}$ $\left(\frac{d\Omega}{dt}\right)_{ia} = -\omega_0^2 \times \sin\left(\theta_i + \frac{j_1}{2}\right) = \frac{k_2}{h}$
B	$\begin{cases} t_i + \frac{h}{2} \\ \theta_i + \left(\frac{d\theta}{dt}\right)_{ia} \times \frac{h}{2} = \theta_i + \frac{j_2}{2} \\ \Omega_i + \left(\frac{d\Omega}{dt}\right)_{ia} \times \frac{h}{2} = \Omega_i + \frac{k_2}{2} \end{cases}$

	$\left(\frac{d\theta}{dt}\right)_{ib} = \Omega_i + \frac{k_2}{2} = \frac{j_3}{h}$ $\left(\frac{d\Omega}{dt}\right)_{ib} = -\omega_0^2 \times \sin\left(\theta_i + \frac{j_2}{2}\right) = \frac{k_3}{h}$
C	$\begin{cases} \quad t_i + h \\ \theta_i + \left(\frac{d\theta}{dt}\right)_{ib} \times h = \theta_i + j_3 \\ \Omega_i + \left(\frac{d\Omega}{dt}\right)_{ib} \times h = \Omega_i + k_3 \end{cases}$ $\left(\frac{d\theta}{dt}\right)_{ic} = \Omega_i + k_3 = \frac{j_4}{h}$ $\left(\frac{d\Omega}{dt}\right)_{ic} = -\omega_0^2 \sin(\theta_i + j_3) = \frac{k_4}{h}$

Soit :

$$j_1 = \Omega_t \times dt$$

$$k_1 = \left(-\omega_0^2 \times \sin\theta_t\right) \times dt$$

$$j_2 = \left[\Omega_t + \frac{k_1}{2}\right] \times dt$$

$$k_2 = \left(-\omega_0^2 \times \sin\left[\theta_t + \frac{j_1}{2}\right]\right) \times dt$$

$$j_3 = \left[\Omega_t + \frac{k_2}{2}\right] \times dt$$

$$k_3 = \left(-\omega_0^2 \times \sin\left[\theta_t + \frac{j_2}{2}\right]\right) \times dt$$

$$j_4 = [\Omega_t + k_3] \times dt$$

$$k_4 = \left(-\omega_0^2 \times \sin[\theta_t + j_3]\right) \times dt$$

$$\Omega(t + dt) = \Omega_t + \frac{1}{6}\left[k_1 + 2(k_2 + k_3) + k_4\right]$$

$$\theta(t + dt) = \theta_t + \frac{1}{6}\left[j_1 + 2(j_2 + j_3) + j_4\right]$$

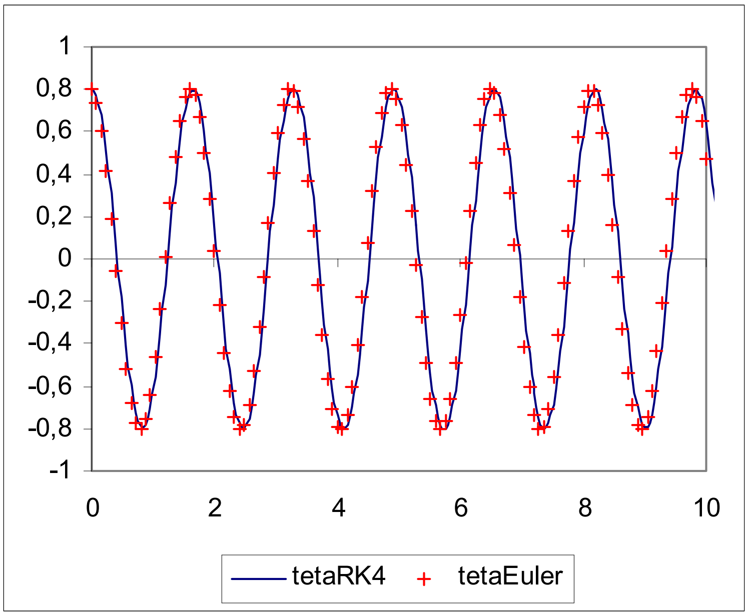
	A	B	C
9	t	teta	omega
10	0	=teta0	=omega0
11	=A10+dt	=B10+1/6*(E10+2*G10+2*I10+K10)	=C10+1/6*(D10+2*F10+2*H10+J10)
12	=A11+dt	=B11+1/6*(E11+2*G11+2*I11+K11)	=C11+1/6*(D11+2*F11+2*H11+J11)
13	=A12+dt	=B12+1/6*(E12+2*G12+2*I12+K12)	=C12+1/6*(D12+2*F12+2*H12+J12)
14	=A13+dt	=B13+1/6*(E13+2*G13+2*I13+K13)	=C13+1/6*(D13+2*F13+2*H13+J13)
15	=A14+dt	=B14+1/6*(E14+2*G14+2*I14+K14)	=C14+1/6*(D14+2*F14+2*H14+J14)

	D	E
9	k1	j1
10	=-dt*w02_*SIN(teta)	=dt*omega
11	=-dt*w02_*SIN(teta)	=dt*omega
12	=-dt*w02_*SIN(teta)	=dt*omega
13	=-dt*w02_*SIN(teta)	=dt*omega
14	=-dt*w02_*SIN(teta)	=dt*omega
15	=-dt*w02_*SIN(teta)	=dt*omega

	F	G
9	k2	j2
10	=-dt*w02_*SIN(teta+j1_/2)	=dt*(omega+k1_/2)
11	=-dt*w02_* SIN(teta+j1_/2)	=dt*(omega+k1_/2)
12	=-dt*w02_* SIN(teta+j1_/2)	=dt*(omega+k1_/2)
13	=-dt*w02_* SIN(teta+j1_/2)	=dt*(omega+k1_/2)
14	=-dt*w02_* SIN(teta+j1_/2)	=dt*(omega+k1_/2)
15	=-dt*w02_* SIN(teta+j1_/2)	=dt*(omega+k1_/2)

	H	I
9	k3	j3
10	=-dt*w02_*SIN(teta+j2_/2)	=dt*(omega+k2_/2)
11	=-dt*w02_* SIN(teta+j2_/2)	=dt*(omega+k2_/2)
12	=-dt*w02_* SIN(teta+j2_/2)	=dt*(omega+k2_/2)
13	=-dt*w02_* SIN(teta+j2_/2)	=dt*(omega+k2_/2)
14	=-dt*w02_* SIN(teta+j2_/2)	=dt*(omega+k2_/2)
15	=-dt*w02_* SIN(teta+j2_/2)	=dt*(omega+k2_/2)

	J	K
9	k4	j4
10	=-dt*w02_*SIN(teta+j3_)	=dt*(omega+k3_)
11	=-dt*w02_* SIN(teta+j3_)	=dt*(omega+k3_)
12	=-dt*w02_* SIN(teta+j3_)	=dt*(omega+k3_)
13	=-dt*w02_* SIN(teta+j3_)	=dt*(omega+k3_)
14	=-dt*w02_* SIN(teta+j3_)	=dt*(omega+k3_)
15	=-dt*w02_* SIN(teta+j3_)	=dt*(omega+k3_)

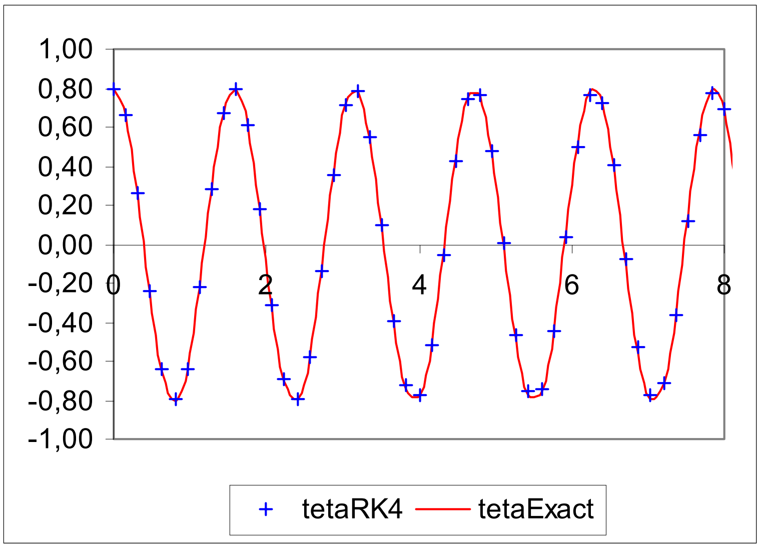
θ_0 = 0,80 rad $\Omega_{(t=0)}$=0,15 rad/s dt = 0,08 s ω_0^2=16 rad/s^2	

Comparer les résultats obtenus par la méthode RK4 et la méthode d'EULER.

Exercice RK4_5 : Pendule simple

On se reportera à l'exercice RK4_4 pour la mise en équation ($\sin\theta \approx \theta$) :

$$\theta'' + \omega_0^2 \times \theta = 0$$

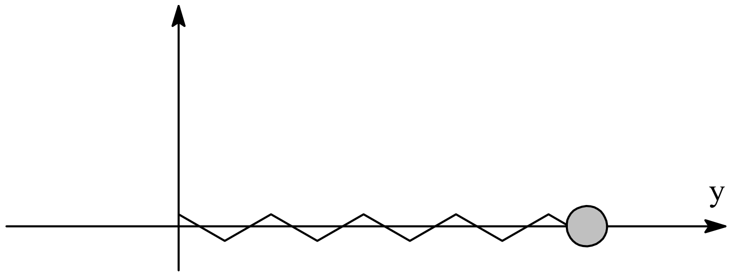
$\theta'' = -\omega_0^2\theta$ θ_0 = 0,80 rad $\Omega_{(t=0)}$ = 0,15 rad/s dt = 0,16 s ω_0^2=16 rad/s^2	

Comparer les résultats obtenus par la méthode RK4 et la solution analytique :

teta =teta$_0$/COS(ATAN(-omega$_0$/w$_0$/teta$_0$))*COS(w$_0$*t+ATAN(-omega$_0$/w$_0$/teta$_0$))

Exercice RK4 6 :Oscillateur amorti

$$y''+2\alpha y'+\omega_0^2 y = 0$$



On pose : $\frac{dy}{dt} = v$

L'équation ci-dessus devient :

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = -2\alpha v - \omega_0^2 y$$

Méthode RK4 :

	$\begin{cases} t_i \\ y_i \\ v_i \end{cases}$ $\left(\frac{dy}{dt}\right)_i = v_i = \frac{j_1}{h}$ $\left(\frac{dv}{dt}\right)_i = -2\alpha \times v_i - \omega_0^2 \times y_i = \frac{k_1}{h}$
A	$\begin{cases} t_i + \frac{h}{2} \\ y_i + \left(\frac{dy}{dt}\right)_i \times \frac{h}{2} = y_i + \frac{j_1}{2} \\ v_i + \left(\frac{dv}{dt}\right)_i \times \frac{h}{2} = v_i + \frac{k_1}{2} \end{cases}$ $\left(\frac{dy}{dt}\right)_{ia} = v_i + \frac{k_1}{2} = \frac{j_2}{h}$ $\left(\frac{dv}{dt}\right)_{ia} = -2\alpha \times \left(v_i + \frac{k_1}{2}\right) - \omega_0^2 \times \left(y_i + \frac{j_1}{2}\right) = \frac{k_2}{h}$

B	$\begin{cases} t_i + \frac{h}{2} \\ y_i + \left(\frac{dy}{dt}\right)_{ia} \times \frac{h}{2} = y_i + \frac{j_2}{2} \\ v_i + \left(\frac{dv}{dt}\right)_{ia} \times \frac{h}{2} = v_i + \frac{k_2}{2} \end{cases}$ $\left(\frac{dy}{dt}\right)_{ib} = v_i + \frac{k_2}{2} = \frac{j_3}{h}$ $\left(\frac{dv}{dt}\right)_{ib} = -2\alpha \times \left(v_i + \frac{k_2}{2}\right) - \omega_0^2 \times \left(y_i + \frac{j_2}{2}\right) = \frac{k_3}{h}$
C	$\begin{cases} t_i + h \\ y_i + \left(\frac{dy}{dt}\right)_{ib} \times h = y_i + j_3 \\ v_i + \left(\frac{dv}{dt}\right)_{ib} \times h = v_i + k_3 \end{cases}$ $\left(\frac{dy}{dt}\right)_{ic} = v_i + k_3 = \frac{j_4}{h}$ $\left(\frac{dv}{dt}\right)_{ic} = -2\alpha \times (v_i + k_3) - \omega_0^2 (y_i + j_3) = \frac{k_4}{h}$

Soit :

$$j_1 = v_t \times dt$$

$$k_1 = \left(-2\alpha \times v_t - \omega_0^2 \times y_t\right) \times dt$$

$$j_2 = \left[v_t + \frac{k_1}{2}\right] \times dt$$

$$k_2 = \left(-2\alpha \times \left[v_t + \frac{k_1}{2}\right] - \omega_0^2 \times \left[y_t + \frac{j_1}{2}\right]\right) \times dt$$

$$j_3 = \left[v_t + \frac{k_2}{2}\right] \times dt$$

$$k_3 = \left(-2\alpha \times \left[v_t + \frac{k_2}{2}\right] - \omega_0^2 \times \left[y_t + \frac{j_2}{2}\right]\right) \times dt$$

$$j_4 = \left[v_t + k_3\right] \times dt$$

$$k_4 = \left(-2\alpha \times [v_t + k_3] - \omega_0^2 \times [y_t + j_3]\right) \times dt$$

$$v(t + dt) = v_t + \frac{1}{6}\left[k_1 + 2(k_2 + k_3) + k_4\right]$$

$$y(t + dt) = y_t + \frac{1}{6}\left[j_1 + 2(j_2 + j_3) + j_4\right]$$

alpha	1,00	s^{-1}
w02	50,0	s^{-2}
dt	0,10	s

v0	0,00	$m.s^{-1}$
y0	0,05	m

	A	B	C
	x	j1	k1
7	0	=dt*J7	=-dt*(2*alpha*v+w02_*y)
8	=A7+dt	=dt*J8	=-dt*(2*alpha*v+w02_*y)
9	=A8+dt	=dt*J9	=-dt*(2*alpha*v+w02_*y)

	D	E	F
	j2	k2	j3
7	=dt*(v+k1_/2)	=-dt*(2*alpha*(v+k1_/2)+w02_*(y+j1_/2))	=dt*(v+k2_/2)
8	=dt*(v+k1_/2)	=-dt*(2*alpha*(v+k1_/2)+w02_*(y+j1_/2))	=dt*(v+k2_/2)
9	=dt*(v+k1_/2)	=-dt*(2*alpha*(v+k1_/2)+w02_*(y+j1_/2))	=dt*(v+k2_/2)

	G	H	I
	k3	j4	k4
7	=-dt*(2*alpha*(v+k2_/2)+w02_*(y+j2_/2))	=dt*(v+k3_)	=-dt*(2*alpha*(v+k3_)+w02_*(y+j3_))
8	=-dt*(2*alpha*(v+k2_/2)+w02_*(y+j2_/2))	=dt*(v+k3_)	=-dt*(2*alpha*(v+k3_)+w02_*(y+j3_))
9	=-dt*(2*alpha*(v+k2_/2)+w02_*(y+j2_/2))	=dt*(v+k3_)	=-dt*(2*alpha*(v+k3_)+w02_*(y+j3_))

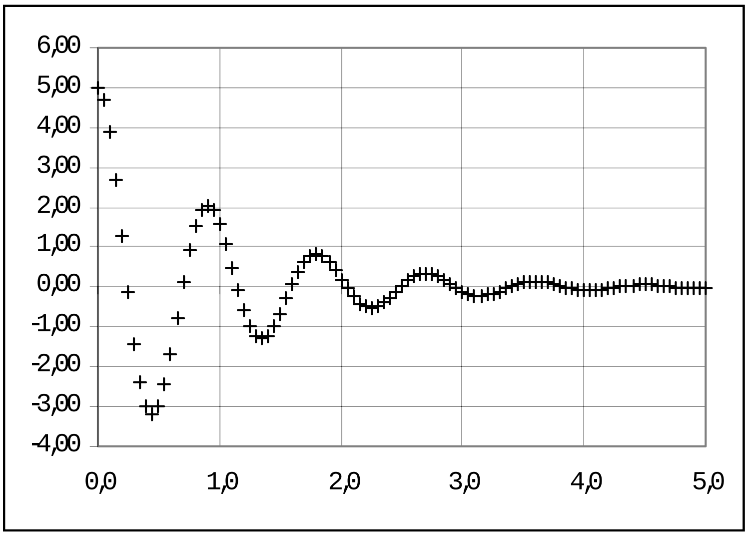
	J	K	L	M
	v	y	k	j
7	=v0	=y0	=1/6*(k1_+2*k2_+2*k3_+k4_)	=1/6*(j1_+2*j2_+2*j3_+j4_)
8	=J7+L7	=K7+M7	=1/6*(k1_+2*k2_+2*k3_+k4_)	=1/6*(j1_+2*j2_+2*j3_+j4_)
9	=J8+L8	=K8+M8	=1/6*(k1_+2*k2_+2*k3_+k4_)	=1/6*(j1_+2*j2_+2*j3_+j4_)

Différents régimes :

Régime pseudo-périodique

$\alpha^2 - \omega_0^2 < 0$

y(0) = 5 cm
v(0) = 0
$\alpha = 1$
$\omega_0^2=50$

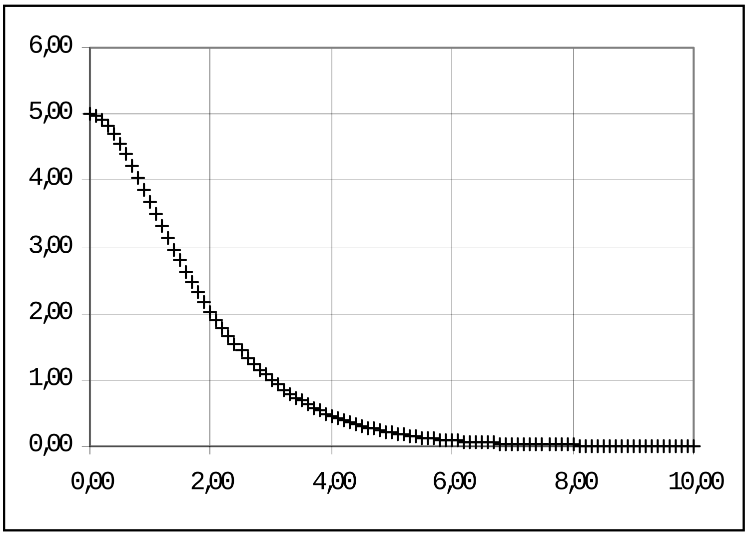


Régime critique

$\alpha^2 - \omega_0^2 = 0$

y(0) = 5 cm
v(0) = 0
$\alpha = 1$
$\omega_0^2=1$

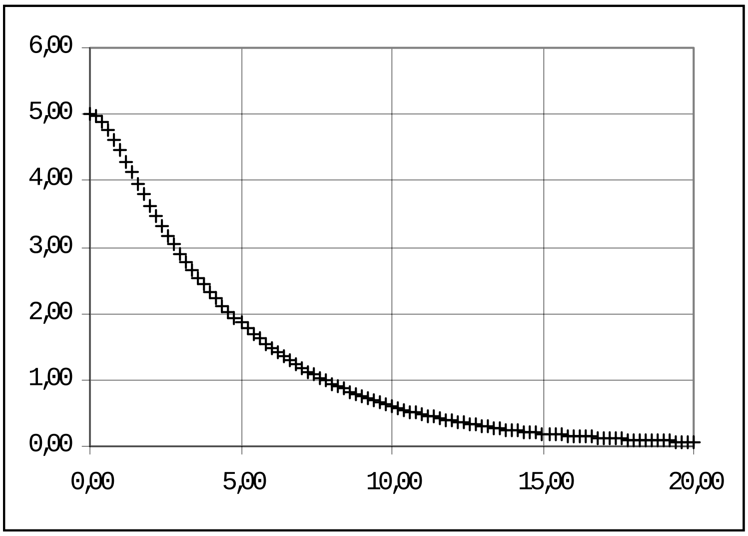
Le retour à l'équilibre s'effectue plus rapidement que dans le régime apériodique



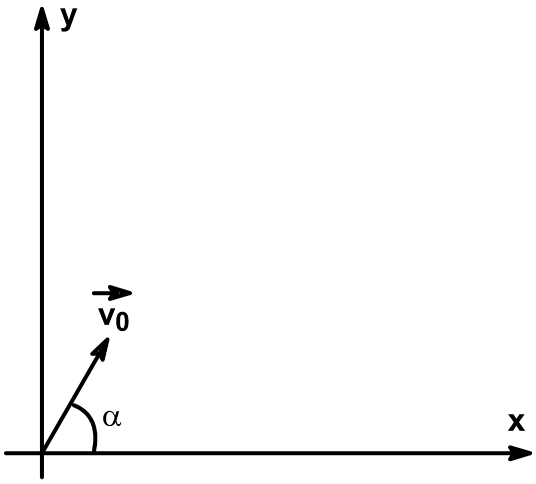
Régime apériodique

$\alpha^2 - \omega_0^2 > 0$

y(0) = 5 cm
v(0) = 0
$\alpha = 1$
$\omega_0^2=0,4$



Exercice RK4 7 : Mouvement d'un projectile avec frottements



$$m\vec{a} = m\vec{g} + \vec{R_f}$$

$\vec{R_f} = -\beta \times \left(V_x^2 \vec{i} + \frac{V_y^3}{|V_y|} \vec{j} \right)$. L'expression $\frac{V_y^3}{|V_y|}$ permet de tenir compte de la valeur algébrique.

$$\begin{cases} \dfrac{d^2x}{dt^2} = -\beta \times V_x^2 \\ \dfrac{d^2y}{dt^2} = -g - \beta \times \dfrac{V_y^3}{|V_y|} \end{cases}$$

<u>Méthode RK4</u>

Pour la projection de l'équation différentielle sur l'axe des x :

$$v_x = \frac{dx}{dt}$$

$$\frac{dv_x}{dt} = -\beta \times V_x^2$$

A partir des conditions initiales : t=0 ; y(0)=0; x(0) =0; $v_x(0)$, $v_y(0)$, on calcule de proche en proches les valeurs suivantes :

P_i	$\begin{cases} t_i \\ x_i \\ (v_x)_i \end{cases}$ $\left(\frac{dx}{dt}\right)_i = (v_x)_i = \frac{j_{x1}}{h}$ $\left(\frac{dv_x}{dt}\right)_i = -\beta \times (v_x)_i^2 = \frac{k_{x1}}{h}$
A	$\begin{cases} t_i + \frac{h}{2} \\ x_i + \left(\frac{dx}{dt}\right)_i \times \frac{h}{2} = x_i + \frac{j_{x1}}{2} \\ (v_x)_i + \left(\frac{dv_x}{dt}\right)_i \times \frac{h}{2} = (v_x)_i + \frac{k_{x1}}{2} \end{cases}$ $\left(\frac{dx}{dt}\right)_{ia} = (v_x)_i + \frac{k_{x1}}{2} = \frac{j_{x2}}{h}$ $\left(\frac{dv_x}{dt}\right)_{ia} = -\beta \times \left((v_x)_i + \frac{k_{x1}}{2}\right)^2 = \frac{k_{x2}}{h}$

B	$$\begin{cases} t_i + \frac{h}{2} \\ x_i + \left(\frac{dx}{dt}\right)_{ia} \times \frac{h}{2} = x_i + \frac{j_{x2}}{2} \\ (v_x)_i + \left(\frac{dv_x}{dt}\right)_{ia} \times \frac{h}{2} = (v_x)_i + \frac{k_{x2}}{2} \end{cases}$$ $$\left(\frac{dx}{dt}\right)_{ib} = (v_x)_i + \frac{k_{x2}}{2} = \frac{j_{x3}}{h}$$ $$\left(\frac{dv_x}{dt}\right)_{ib} = -\beta \times \left((v_x)_i + \frac{k_{x2}}{2}\right)^2 = \frac{k_{x3}}{h}$$
C	$$\begin{cases} t_i + h \\ x_i + \left(\frac{dx}{dt}\right)_{ib} \times h = x_i + j_{x3} \\ (v_x)_i + \left(\frac{dv_x}{dt}\right)_{ib} \times h = (v_x)_i + k_{x3} \end{cases}$$ $$\left(\frac{dx}{dt}\right)_{ic} = (v_x)_i + k_{x3} = \frac{j_{x4}}{h}$$ $$\left(\frac{dv_x}{dt}\right)_{ic} = -\beta\left((v_x)_i + k_{x3}\right)^2 = \frac{k_{x4}}{h}$$
P_{i+1}	$$(v_x)_{i+1} = (v_x)_i + \frac{1}{6}\left[k_{x1} + 2(k_{x2} + k_{x3}) + k_{x4}\right]_i$$ $$x_{i+1} = x_i + \frac{1}{6}\left[j_{x1} + 2(j_{x2} + j_{x3}) + j_{x4}\right]_i$$

Pour la projection de l'équation différentielle sur l'axe des y :

$$v_y = \frac{dy}{dt}$$

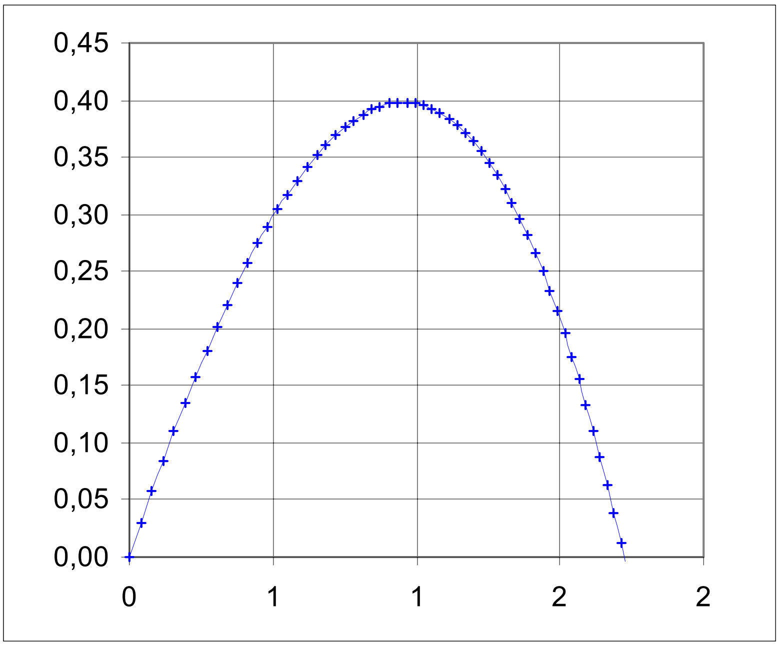
$$\frac{dv_y}{dt} = -g - \beta \times \frac{V_y^3}{|v_y|}$$

A partir des conditions initiales : t=0 ; y(0)=0; x(0) =0; $v_x(0)$, $v_y(0)$, on calcule de proche en proches les valeurs suivantes :

P_i	$\begin{cases} t_i \\ y_i \\ (v_y)_i \end{cases}$ $(v_y)_i = \left(\frac{dy}{dt}\right)_i = \frac{j_{y1}}{h}$ $\left(\frac{dv_y}{dt}\right)_i = -g - \beta \times \frac{(v_y)_i^3}{\lvert(v_y)_i\rvert} = \frac{k_{y1}}{h}$
A	$\begin{cases} t_i + \frac{h}{2} \\ y_i + \left(\frac{dy}{dt}\right)_i \times \frac{h}{2} = y_i + \frac{j_{y1}}{2} \\ (v_y)_i + \left(\frac{dv_y}{dt}\right)_i \times \frac{h}{2} = (v_y)_i + \frac{k_{y1}}{2} \end{cases}$ $\left(\frac{dy}{dt}\right)_{ia} = (v_y)_i + \frac{k_{y1}}{2} = \frac{j_{y2}}{h}$ $\left(\frac{dv_y}{dt}\right)_{ia} = -g - \beta \times \frac{\left((v_y)_i + \frac{k_{y1}}{2}\right)^3}{\left\lvert(v_y)_i + \frac{k_{y1}}{2}\right\rvert} = \frac{k_{y2}}{h}$

B	$$\begin{cases} t_i + \frac{h}{2} \\ y_i + \left(\frac{dy}{dt}\right)_{ia} \times \frac{h}{2} = y_i + \frac{j_{y2}}{2} \\ (v_y)_i + \left(\frac{dv_y}{dt}\right)_{ia} \times \frac{h}{2} = (v_y)_i + \frac{k_{y2}}{2} \end{cases}$$ $$\left(\frac{dy}{dt}\right)_{ib} = (v_y)_i + \frac{k_{y2}}{2} = \frac{j_{y3}}{h}$$ $$\left(\frac{dv_y}{dt}\right)_{ib} = -g - \beta \times \frac{\left((v_y)_i + \frac{k_{y2}}{2}\right)^3}{\left\|(v_y)_i + \frac{k_{y2}}{2}\right\|} = \frac{k_{y3}}{h}$$
C	$$\begin{cases} t_i + h \\ y_i + \left(\frac{dy}{dt}\right)_{ib} \times h = y_i + j_{y3} \\ (v_y)_i + \left(\frac{dv_y}{dt}\right)_{ib} \times h = (v_y)_i + k_{y3} \end{cases}$$ $$\left(\frac{dy}{dt}\right)_{ic} = (v_y)_i + k_{y3} = \frac{j_{y4}}{h}$$ $$\left(\frac{dv_y}{dt}\right)_{ic} = -g - \beta \frac{((v_y)_i + k_{y3})^3}{\left\|(v_y)_i + k_{y3}\right\|} = \frac{k_{y4}}{h}$$
P_{i+1}	$$(v_y)_{i+1} = v_{y_i} + \frac{1}{6}\left[k_{y1} + 2(k_{y2} + k_{y3}) + k_{y4}\right]_i$$ $$y_{i+1} = y_i + \frac{1}{6}\left[j_{y1} + 2(j_{y2} + j_{y3}) + j_{y4}\right]_i$$

dt	0,01	S
g	10,0	$m.s^{-2}$
B	0,30	m^{-1}
vx0	4,00	$m.s^{-1}$
vy0	3,00	$m.s^{-1}$



y=f(x)